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On the Minimal Norms of Polynomial Projections

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In this paper the asymptotically sharp lower bound $(4/\pi^2)(\ln n - \ln \ln n)$ for the norms of linear projections from $C[-1, 1]$ onto the polynomials of nth degree is proved. As a consequence, we obtain the asymptotical minimality for some sequences of projections and particularly for the Chebyshev partial sum operators.

1. INTRODUCTION

When approximating continuous functions on the interval $[-1, 1]$, polynomial projections are used frequently. Such projections, L_n, are bounded linear operators mapping $C[-1, 1]$ onto the subspace, Π_n, of all algebraic polynomials of degree less than or equal to n, and having the property that $L_n[p] = p$ for all $p \in \Pi_n$.

The error of this approximation can be estimated using the Lesbesgue inequality,

$$\|L_n[f] - f\|_\infty \leqslant (1 + \|L_n\|) \cdot E_n[f],$$

where

$$\|L_n\| = \sup_{\|f\|_\infty \leqslant 1} \|L_n[f]\|_\infty$$

is the norm of L_n and $E_n[f]$ denotes the error of the best approximation of f by elements of Π_n. The quality of a projection therefore depends on its norm.

Since it seems to be a very hard problem to find minimal projections $L_n^{\min}$, i.e., projections onto Π_n with smallest possible norms (they are still unknown unless $n = 1$), we at least would like to know projections whose norms differ only a little from $\|L_n^{\min}\|$. For this purpose, we need lower

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bounds which enable us to prove the asymptotical minimality for some sequences $(L_n)_{n\in\mathbb{N}}$, i.e., the property

$$\lim_{n\to\infty}\frac{\|L_n\|}{\|L_n^{\min}\|}=1.$$

Until now, for arbitrary n, only the inequality (cf. [2, p. 214; 4])

$$\|S_n\|-A\geqslant\|L_n^{\min}\|\geqslant\frac{1}{2}\|S_0+S_n\|=\frac{2}{\pi^2}\ln n+O(1) \tag{1}$$

has been known, where $A>0$ and S_n denotes the Chebyshev partial sum operators (cf. [2]) with norms (see [8])

$$\|S_n\|=\frac{4}{\pi^2}\ln(2n+1)+\gamma+\rho_n,$$
$$\text{where}\quad \gamma=0.989431\ldots\text{ and } 0\leqslant\rho_n\leqslant\frac{0.012}{(2n+1)^2}. \tag{2}$$

Although it has been regarded as an important question to diminish the coefficient $4/\pi^2$ of $\ln n$ in the asymptotical evaluation for a sequence $(\|L_n\|)_{n\in\mathbb{N}}$ (cf. [4]) and therefore several projections with small norms have been examined in the past (cf. [3, 4]), the upper bound $\|S_n\|$ for $\|L_n^{\min}\|$ could be improved [4] only by a constant summand as stated above. We therefore might expect that $4/\pi^2$ is the best possible coefficient. Indeed, the inequality in the theorem below implies

$$\lim_{n\to\infty}\frac{\|L_n^{\min}\|}{\ln n}=\frac{4}{\pi^2}.$$

Hence, in Section 3, we obtain some sequences of asymptotically minimal projections.

2. The Lower Bound

First, we introduce the following notation:

$$\mathscr{P}_n:=\{L\mid L\text{ is a projection from }C[-1,1]\text{ onto }\Pi_n\}$$
$$C^e[0,\pi]:=\{f\mid f\text{ is a continuous, even, }2\pi\text{-periodic function}\}$$
$$\mathscr{T}_n^e:=\left\{c\in C^e[0,\pi]\;\middle|\;c(x)=\sum_{\nu=0}^{n}a_\nu\cos\nu x\right\}$$
$$\mathscr{H}_n:=\{H\mid H\text{ is a projection from }C^e[0,\pi]\text{ onto }\mathscr{T}_n^e\}.$$

Furthermore, let T_λ and S_n^c be the operators defined by

$$T_\lambda[f](x) := f(x+\lambda)$$

and

$$\begin{aligned} S_n^c[f](x) &:= \frac{2}{\pi} \sum_{k=0}^{n}{}' \cos kx \cdot \int_0^\pi f(t) \cos kt \, dt \\ &= \frac{1}{\pi} \int_0^\pi f(t) \{ D_n(x+t) + D_n(x-t) \} \, dt, \end{aligned}$$

where

$$D_n(u) = \tfrac{1}{2} \sin(n+\tfrac{1}{2})u \cdot \csc \tfrac{1}{2}u.$$

($\sum'$ denotes that the first summand should be halved.)

Our main result is

THEOREM. *For every $n \geqslant 2$, the norms of projections $L_n \in \mathscr{P}_n$ are bounded as follows:*

$$\|L_n\| \geqslant \frac{4}{\pi^2} (\ln n - \ln \ln n). \tag{3}$$

The proof of the theorem will be a refinement of the well known proof of the lower estimate in (1). We therefore use the following Lemmata (cf. [2, p. 214; 5])

LEMMA 1. *For every projection in $\mathscr{P}_n$ there exists one in $\mathscr{H}_n$ having the same norm and vice versa.*

LEMMA 2. *Let H be an arbitrary projection in $\mathscr{H}_n$, then*

$$S_0^c + S_n^c = \frac{1}{2\pi} \int_{-\pi}^{\pi} T_\lambda H(T_\lambda + T_{-\lambda}) \, d\lambda.$$

LEMMA 3. *For every projection $H \in \mathscr{H}_n$ and every $\delta > 0$, there exists a projection in $\mathscr{H}_n$ with finite carrier whose norm is bounded by $\|H\| + \delta$.*

Proof of the Theorem. The lower bounds are almost trivial for $n \leqslant 44$, since they are less than 1 in these cases.

Now, let $n > 44$: Lemma 1 and 3 imply that the inequality (3) must be proved for projections in $\mathscr{H}_n$ with finite carrier. We therefore can assume $H[f] := \sum_{\nu=1}^{m} f(t_\nu)\, l_\nu$, where $t_\nu \in [0, \pi]$ and $l_\nu \in \mathscr{T}_n^c$.

Let g be the even, 2π-periodic function defined on $[0, \pi]$ by

$$g(t) := \begin{cases} \operatorname{sgn} D_n\left(\frac{\pi}{2} - t\right), & \text{if } \quad t \in \left[\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon\right]; \\ 0, & \text{otherwise;} \end{cases}$$

with an arbitrary $\varepsilon \in [0, \pi/4]$, and let

$$g_\lambda := (T_\lambda + T_{-\lambda})[g].$$

One verifies readily that

$$S_0^c[g]\left(\frac{\pi}{2}\right) = \frac{1}{\pi} \int_0^\pi g(t)\, dt \geqslant 0,$$

and so, since the norm of S_n is $(2/\pi) \int_0^\pi |D_n(t)|\, dt$ (cf. [2, p. 212]), we obtain

$$\begin{aligned} (S_0^c + S_n^c)[g]\left(\frac{\pi}{2}\right) &\geqslant \frac{1}{\pi} \int_{\pi/2 - \varepsilon}^{\pi/2 + \varepsilon} g(t) \left\{D_n\left(\frac{\pi}{2} + t\right) + D_n\left(\frac{\pi}{2} - t\right)\right\} dt \\ &= \frac{2}{\pi} \int_0^\varepsilon |D_n(t)|\, dt + \frac{2}{\pi} \int_0^\varepsilon g\left(t + \frac{\pi}{2}\right) D_n(\pi + t)\, dt \\ &\geqslant \frac{2}{\pi} \int_0^\pi |D_n(t)|\, dt - \frac{2}{\pi} \int_\varepsilon^\pi |D_n(t)|\, dt - \frac{\varepsilon}{\pi} \sec \frac{\varepsilon}{2} \\ &= \|S_n\| - \frac{2}{\pi} R_\varepsilon - \frac{\varepsilon}{\pi} \sec \frac{\varepsilon}{2}, \end{aligned}$$

where

$$R_\varepsilon = \int_{\varepsilon/2}^{\pi/2} \left|\frac{\sin(2n+1)t}{\sin t}\right| dt.$$

We define $\psi_\nu := \varepsilon/2 + \nu\pi/(2n+1)$ and choose μ such that $\psi_\mu < \pi/2 \leqslant \psi_{\mu+1}$. The cosecant function is monotonically decreasing in $[0, \pi/2]$, so that

$$\begin{aligned} R_\varepsilon &\leqslant \sum_{\nu=0}^{\mu} \csc \psi_\nu \cdot \int_{\psi_\nu}^{\psi_{\nu+1}} |\sin(2n+1)t|\, dt \\ &= \frac{2}{2n+1} \sum_{\nu=0}^{\mu} \csc \psi_\nu \\ &\leqslant \frac{2}{2n+1} \csc \frac{\varepsilon}{2} + \frac{2}{\pi} \int_{\varepsilon/2}^{\pi/2} \csc x\, dx \\ &\leqslant \frac{2}{2n+1} \csc \frac{\varepsilon}{2} + \frac{2}{\pi} \ln \frac{4}{\varepsilon}. \end{aligned}$$

Furthermore, the equation

$$\{T_\lambda H(T_\lambda + T_{-\lambda})\}[g]\left(\frac{\pi}{2}\right) = \{T_\lambda H\}[g_\lambda]\left(\frac{\pi}{2}\right)$$
$$= \left\{T_\lambda \sum_{\nu=1}^{m} g_\lambda(t_\nu)\, l_\nu\right\}\left(\frac{\pi}{2}\right)$$
$$= \sum_{\nu=1}^{m} g_\lambda(t_\nu)\, l_\nu\left(\frac{\pi}{2}+\lambda\right)$$

gives rise to the inequality

$$(S_0^\varepsilon + S_n^\varepsilon)[g]\left(\frac{\pi}{2}\right) \leqslant \frac{1}{2\pi}\sum_{\nu=1}^{m}\int_{-\pi}^{\pi} |g_\lambda(t_\nu)| \cdot \left| l_\nu\left(\frac{\pi}{2}+\lambda\right)\right| d\lambda.$$

$|g_\lambda(t_\nu)|$ can only exceed 1 (i.e., be equal to 2), if $t_\nu + \lambda \in [(i-\frac{1}{2})\pi - \varepsilon, (i-\frac{1}{2})\pi + \varepsilon]$ and $t_\nu - \lambda \in [(j-\frac{1}{2})\pi - \varepsilon, (j-\frac{1}{2})\pi + \varepsilon]$; $i, j \in \{0, 1, 2\}$ simultaneously, and hence if

$$\lambda \in L := [-\pi, \pi] \cap \left\{x \,\middle|\, \left|x - \frac{\nu\pi}{2}\right| \leqslant \varepsilon \text{ for an integer } \nu\right\}.$$

Since L has measure 8ε,

$$(S_0^\varepsilon + S_n^\varepsilon)[g]\left(\frac{\pi}{2}\right) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \Lambda\left(\frac{\pi}{2}+\lambda\right) d\lambda + \frac{1}{2\pi}\int_L \Lambda\left(\frac{\pi}{2}+\lambda\right) d\lambda$$
$$\leqslant \left(1 + \frac{4\varepsilon}{\pi}\right) \cdot \|H\|,$$

where $\Lambda = \sum_{\nu=1}^{m} |l_\nu|$ denotes the even, 2π-periodic Lebesgue function of H, (for which the well known relation $\|H\| = \|\Lambda\|_\infty$ holds). It follows from the given inequalities that

$$\|H\| \geqslant \frac{1}{1+4\varepsilon/\pi}\left(\|S_n\| - \frac{4}{\pi^2}\ln\frac{4}{\varepsilon} - \frac{4}{(2n+1)\pi}\csc\frac{\varepsilon}{2} - \frac{\varepsilon}{\pi}\sec\frac{\varepsilon}{2}\right). \tag{4}$$

Choosing $\varepsilon := \pi/(4\ln n)$, we obtain

$$\|L_n\| \geqslant \frac{\ln n}{1+\ln n}\left(\|S_n\| - \frac{4}{\pi^2}\ln\left(\frac{16}{\pi}\ln n\right)\right.$$
$$\left. - \frac{1}{4\ln n}\sec\frac{\pi}{8\ln n} - \frac{4}{(2n+1)\pi}\csc\frac{\pi}{8\ln n}\right).$$

The theorem follows now using the inequality $\|S_n\| \geqslant (4/\pi^2)\ln n + 1.27$, which is a simple consequence of (2). ∎

Remark 1. Strictly speaking, the special choice of the discontinuous function g in the proof of the theorem is not correct, but we can modify g on a set of measure arbitrarily close to 0 without changing its norm, such that the new function is continuous and takes the value 0 in the same intervals as g. The upper estimate of $S_0^c + S_n^c$ does not change, while the lower bound is reduced by an arbitrarily small amount, so that the inequality (3) still remains valid.

Remark 2. An elementary computation shows that, using another ε in (4), the lower bound of the theorem can only be improved by a summand of the order $o(\ln \ln n)$.

3. Asymptotically Minimal Projections

A simple consequence of the theorem and Eq. (2) is

Corollary 1. *The Chebyshev partial sum operators are asymptotically minimal.*

The purpose of linear polynomial projections is having a simple method for approximating functions. However, the computation of S_n requires the knowledge of the values of $n+1$ integrals, which cannot always be assumed. Usually, we can use only function values, thus we need projections with finite carrier and in particular with a small carrier. A lower bound for the number of required function values for projections is $n+1$, which is taken by interpolation operators. Since it is impossible to find a sequence of interpolation operators being asymptotically minimal (sharp lower bounds for those operators are given in Vértesi [7]), we search for asymptotically minimal projections with asymptotically finite carrier, i.e., the ratio of the number of required function values and $n+1$ will tend to 1.

Projections, having small carriers as well as small norms have been defined by Lewanowicz [3] as follows:

$$S_n^{(m)}[f] = \sum_{k=0}^{n}{}' \alpha_k^{(m)}[f]\, T_k; \qquad m \geqslant n,$$

where

$$\alpha_k^{(m)}[f] = \frac{2}{m+1} \sum_{j=0}^{m} f(x_j)\, T_k(x_j), \qquad x_j = \cos \frac{2j+1}{2m+2}\, \pi$$

and T_k denotes the kth Chebyshev polynomial, i.e., the $S_n^{(m)}$ are the orthogonal polynomial expansions with respect to the inner product $(g, h) = [2/(m+1)] \sum_{j=0}^{m} g(x_j)\, h(x_j)$. An important property of those projections is

$$\|S_n^{(m)}[f] - f\|_\infty \leqslant \frac{1}{2^n(n+1)!}\, \|f^{(n+1)}\|_\infty$$

(cf. [1]), where the right-hand side is also the best possible upper bound for $E_n[f]$ in the space $C^{n+1}[-1, 1]$.

Let now α and β be relatively prime numbers with $\alpha > \beta$, and let $m := m_n := \alpha n/\beta + O(1)$. Then it has already been shown that

$$\|S_n^{(m)}\| = \frac{\pi}{2\alpha} \csc \frac{\pi}{2\alpha} \cdot \frac{4}{\pi^2} \ln n + O(1) \tag{5}$$

(cf. [6]). We therefore have

COROLLARY 2. *There exists a sequence of Lewanowicz operators $S_n^{(m)}$ being asymptotically minimal and having asymptotically minimal carrier.*

Proof. Let $m_{n,s} := [(2^s+1)n/2^s]$. According to (5), we can choose n_s such that

$$\|S_n^{(m_{n,s})}\| \leqslant (1+2^{-s}) \cdot \frac{4}{\pi^2} \ln n \qquad \text{for} \quad n > n_s,$$

because $\alpha = 2^s + 1$ and hence $\pi/(2\alpha) \csc[\pi/(2\alpha)] < 1 + 2^{-s}$. Defining

$$m := m_n := \begin{cases} n+1 & \text{if} \quad n \leqslant n_1; \\ m_{n,s} & \text{if} \quad n_s < n \leqslant n_{s+1}, \end{cases}$$

the corollary follows readily. ∎

Remark 3. Amongst the other type of operators defined in [3] there is also a sequence of asymptotically minimal projections with asymptotically minimal carrier (cf. [6]).

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